Published by Institute of Physics Publishing for SISSA

Received: *January 11, 2007*
Accepted: *February 21, 2007*
Published: *March 15, 2007*

Quantization and holomorphic anomaly

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Abstract: We study wave functions of B-model on a Calabi-Yau threefold in various polarizations.

Keywords: Topological Field Theories, Anomalies in Field and String Theories, Topological Strings.

JHEP03(2007)062

*partly supported by NSF grant No. DMS 0505735.
†partly supported by NSF grant No. DMS 0505735 and 0604552.

Contents

1. Introduction

In present paper, we consider wave functions of B-model on a Calabi-Yau threefold in various polarizations and relations between these wave functions.

One can interpret genus 0 B-model on a Calabi-Yau threefold X as a theory of variations of complex structures. The extended moduli space of complex structures, the space of pairs (complex structure, holomorphic 3-form), can be embedded into the middle-dimensional cohomology $H^3(X, \mathbb{C})$ as a lagrangian submanifold. The B-model for arbitrary genus coupled to gravity (the B-model topological string) can be obtained from genus 0 B-model by means of quantization; the role of Planck constant is played by λ^2 where λ is the string coupling constant. (This is a general statement valid for any topological string. It was derived by Witten [12] from worldsheet calculation of [2].) The partition function of B-model is represented by wave function depending on choices of polarization in $H^3(X, \mathbb{C})$. If the polarization does not depend holomorphically on the points of the moduli space of complex structures, then the dependence of wave function of the points of the moduli space is not necessarily holomorphic. (The $\bar{t}$-dependence is governed by the holomorphic anomaly equation.) This happens, in particular, for a polarization that we call complex hermitian polarization. Other papers use the term "holomorphic polarization" for a complex hermitian polarization in the sense of present paper; we reserve the term "holomorphic polarization" for a polarization that depends holomorphically on the points of the moduli space of complex structures. The holomorphic polarization in our sense was widely used in mirror symmetry; this polarization and its p-adic analog were used to analyze integrality of instanton numbers (genus 0 Gopakumar-Vafa invariants) [8].

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The main goal of present paper is to study wave functions in various polarizations, especially in holomorphic polarization. We believe that Gopakumar-Vafa invariants for any genus can be defined by means of p-adic methods and this definition will have as a consequence integrality of these invariants. The present paper is a necessary first step in the realization of this program. It served as a basis for a conjecture about the structure of Frobenius map on p-adic wave functions formulated in [10]; this conjecture implies integrality of Gopakumar-Vafa invariants.

We begin with a short review of quantization of symplectic vector space (section 2). In section 3, 4 and 5 we use the general results of section 2 to obtain relations between the wave functions of B-model in real, complex hermitian and holomorphic polarizations. In section 6 we compare these wave functions with worldsheet calculations of [2].

The holomorphic anomaly equations were recently studied and applied in [1], [5], [6], [9], [11]. Some of equations in our paper differ slightly from corresponding equations in [1], [11]. However, this difference does not affect any conclusions of these papers.

Acknowledgments

We would like to thank M. Aganagic and M. Kontsevich for helpful discussions.

2. Quantization

We consider a real symplectic vector space V and a symplectic basis of V. (A symplectic structure can be considered as a skew symmetric non-degenerate bilinear form $<\ ,\ >$ on V; we say that $e^\alpha,\ e_\beta,\ \alpha,\beta = 1,\cdots,n = \dim(V)$ is a symplectic basis if $< e^\alpha, e^\beta >=< e_\alpha, e_\beta >= 0$, $< e^\alpha, e_\beta >= \delta^\alpha_\beta$.) It is well known that for every symplectic basis $e = \{e^\alpha, e_\beta\}$, one can construct a Hilbert space $\mathcal{H}_e$; these spaces form a bundle over the space $\mathcal{M}$ of all symplectic bases and one can construct a projectively flat connection on this bundle.[1] The situation does not change if we consider, instead of a real basis in V, a basis $\{e^\alpha, e_\alpha\}$ in the complexification of V requiring that e_α be complex conjugate to e^α.

The picture we described above is the standard picture of quantization of a symplectic vector space. The choice of a basis in V specifies a real polarization; the choice of a basis in its complexification determines a complex polarization. The quantum mechanics lives in Hilbert space of functions depending on $n = \frac{1}{2}\dim V$ variables. To construct this Hilbert space, we should fix a polarization, but Hilbert spaces corresponding to different polarizations can be identified up to a constant factor. In semiclassical approximation, vectors in Hilbert space correspond to lagrangian submanifolds of V.

Let us describe the Hilbert space $\mathcal{H}_e$ for the case when $e = \{e^\alpha, e_\beta\}$ is a symplectic basis of V. An element of V can be represented as a linear combination of vectors e^α, e_β

[1] One says that a connection ∇ on a vector bundle over a space B is projectively flat if $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + C$, where C is a constant depending on X, Y. For an infinite dimensional vector bundle over a compact manifold B with a unitary connection, this means that for every two points $e, \tilde{e}$ of B connected by a continuous path in B, there is an isomorphism between the fibers $\mathcal{H}_e$ and $\mathcal{H}_{\tilde{e}}$ defined up to multiplication by a constant; this isomorphism depends on the homotopy class of the path. We say that a section Φ of the vector bundle is projectively flat if $\nabla_X \Phi = C_X \Phi$ where C_X is a scalar function on the base.

with coefficients x_α, x^β. After quantization, x_α and x^β become self-adjoint operators $\hat{x}_\alpha$, $\hat{x}^\beta$ obeying canonical commutation relations(**CCR**):

$$[\hat{x}_\alpha, \hat{x}_\beta] = [\hat{x}^\alpha, \hat{x}^\beta] = 0, \qquad [\hat{x}_\alpha, \hat{x}^\beta] = \frac{\hbar}{i}\delta_\alpha^\beta \tag{2.1}$$

We define $\mathcal{H}_e$ as the space of irreducible unitary representation of canonical commutation relations.

A (linear) symplectic transformation transforms a symplectic basis $\{e^\alpha, e_\alpha\}$ into symplectic basis $\{\tilde{e}^\alpha, \tilde{e}_\alpha\}$:

$$\begin{aligned} \tilde{e}^\alpha &= M^\alpha_\beta e^\beta + N^{\alpha\beta} e_\beta \\ \tilde{e}_\alpha &= R_{\alpha\beta} e^\beta + S^\beta_\alpha e_\beta. \end{aligned} \tag{2.2}$$

This transformation acts on $\hat{x}_\alpha, \hat{x}^\alpha$ as a canonical transformation, i.e. the new operators $\hat{\tilde{x}}_\alpha, \hat{\tilde{x}}^\alpha$ also obey **CCR**; they are related to $\hat{x}_\alpha, \hat{x}^\alpha$ by the formula:

$$\begin{aligned} \hat{x}^\alpha &= N^{\beta\alpha}\hat{\tilde{x}}_\beta + S^\alpha_\beta \hat{\tilde{x}}^\beta \\ \hat{x}_\alpha &= M^\beta_\alpha \hat{\tilde{x}}_\beta + R_{\beta\alpha}\hat{\tilde{x}}^\beta. \end{aligned} \tag{2.3}$$

It follows from the uniqueness of unitary irreducible representation of **CCR** that there exists a unitary operator T obeying

$$\begin{aligned} \hat{\tilde{x}}^\alpha &= T\hat{x}^\alpha T^{-1}, \\ \hat{\tilde{x}}_\alpha &= T\hat{x}_\alpha T^{-1}. \end{aligned} \tag{2.4}$$

This operator T is defined up to a constant factor relating $\mathcal{H}_e$ and $\mathcal{H}_{\tilde{e}}$. In the case when $\{\tilde{e}^\alpha, \tilde{e}_\alpha\}$ is an infinitesimal variation of $\{e^\alpha, e_\alpha\}$, i.e. $\tilde{e} = e + \delta e$ where

$$\begin{aligned} \delta e^\alpha &= m^\alpha_\beta e^\beta + n^{\alpha\beta} e_\beta, \\ \delta e_\alpha &= r_{\alpha\beta} e^\beta + s^\beta_\alpha e_\beta, \end{aligned} \tag{2.5}$$

we can represent the operator T as $1 + \delta T$, where

$$\delta T = -\frac{1}{2\hbar} n^{\alpha\beta}\hat{x}_\alpha \hat{x}_\beta + \frac{1}{\hbar} m^\beta_\alpha \hat{x}^\alpha \hat{x}_\beta - \frac{1}{2\hbar} r_{\alpha\beta}\hat{x}^\alpha \hat{x}^\beta + C. \tag{2.6}$$

This formula determines a projectively flat connection on the bundle with fibers $\mathcal{H}_e$ and the base consisting of all symplectic bases in V. A quantum state specifies a projectively flat section of this bundle.

The irreducible unitary representation of **CCR** can be realized by operators of multiplication and differentiation on the space of square integrable functions of $x^1, \cdots, x^n$; one can take $\hat{x}^\alpha \Psi = x^\alpha \Psi$ and $\hat{x}_\alpha \Psi = \frac{\hbar}{i}\frac{\partial \Psi}{\partial x^\alpha}$. Then a projectively flat connection takes the form:

$$\delta \Psi = \frac{\hbar}{2} n^{\alpha\beta} \frac{\partial^2 \Psi}{\partial x^\alpha \partial x^\beta} + m^\beta_\alpha x^\alpha \frac{\partial \Psi}{\partial x^\beta} - \frac{1}{2\hbar} r_{\alpha\beta} x^\alpha x^\beta \Psi + C\Psi. \tag{2.7}$$

We will call elements of $\mathcal{H}_e$ and corresponding functions of $x^1, \cdots, x^n$ wave functions.

It is important to notice that in Equation (2.7) instead of square integrable functions, we can consider functions $\Psi(x^1, \cdots, x^n)$ from an almost arbitrary space $\mathcal{E}$; the only essential requirement is that the multiplication by x^α and differentiation with respect to x^α should be defined on a dense subset of $\mathcal{E}$ and transform this set into itself.

Sometimes it is convenient to restrict ourselves to the space of functions of the form $\Psi = \exp(\frac{\Phi}{\hbar})$ where $\Phi = \sum \varphi_n \hbar^n$ is a formal series with respect to $\hbar$ (semiclassical wave functions). Rewriting (2.7) on this space we obtain

$$\delta\Phi = \frac{1}{2} n^{\alpha\beta} \left(\hbar \frac{\partial^2 \Phi}{\partial x^\alpha \partial x^\beta} + \frac{\partial \Phi}{\partial x^\alpha} \frac{\partial \Phi}{\partial x^\beta} \right) + m^\beta_\alpha x^\alpha \frac{\partial \Phi}{\partial x^\beta} - \frac{1}{2} r_{\alpha\beta} x^\alpha x^\beta + \hbar C.$$

Let B be the set of all symplectic bases in the complexification of V. We consider the total space of a bundle over B as the direct product $B \times \mathcal{E}$. One can use Equation (2.7) to define a projectively flat connection on this vector bundle. (The coefficients of infinitesimal variation (2.5) of the basis in V must be real; if we consider $\{e^\alpha, e_\alpha\}$ as a basis of complexification of V, the coefficients of infinitesimal variation obey the same conditions $n^{\alpha\beta} = n^{\beta\alpha}, r_{\alpha\beta} = r_{\beta\alpha}, m^\alpha_\beta + s^\beta_\alpha = 0$, but they can be complex.)

Notice, however that in the real case we are dealing with unitary connection; the operator $T_{e,\tilde{e}}$ that identifies two fibers (up to a constant factor) always exists. In complex case, the equation for projectively flat section can have solutions only over a part of the set of symplectic bases. (Recall that the fibers of our vector bundles are infinite-dimensional.)

It is easy to write down simple formulas for the operator $T_{e,\tilde{e}}$ in the case when $N^{\alpha\beta} = 0$ or $R_{\alpha\beta} = 0$. In the first case we have

$$T_{e,\tilde{e}}(\Psi)(x^\alpha) = \exp\left(-\frac{1}{2\hbar} (RM^{-1})_{\alpha\gamma} x^\alpha x^\gamma \right) \Psi(S^\alpha_\beta x^\beta), \tag{2.8}$$

in the second case

$$T_{e,\tilde{e}}(\Psi)(x^\alpha) = \exp\left(-\frac{\hbar}{2} (MN^T)^{\alpha\gamma} \frac{\partial^2}{\partial x^\alpha \partial x^\gamma} \right) \Psi(S^\alpha_\beta x^\beta). \tag{2.9}$$

Combining Equations (2.8) and (2.9), we obtain an expression for $T_{e,\tilde{e}}$ that is valid when M and S are non-degenerated matrices,

$$T_{e,\tilde{e}}\Psi(x^\alpha) = \exp\left(-\frac{1}{2\hbar} (RM^{-1})_{\alpha\gamma} x^\alpha x^\gamma \right) \left\{ \exp\left(-\frac{\hbar}{2} (MN^T)^{\alpha\gamma} \frac{\partial^2}{\partial x^\alpha \partial x^\gamma} \right) \left[\Psi((M^{-1})^\alpha_\beta x^\beta) \right] \right\}. \tag{2.10}$$

Using the expression (2.10) and Wick's theorem, it is easy to construct diagram techniques to calculate $T_{e,\tilde{e}} e^F$.

Recall that Wick's theorem permits us to represent an expression of the form $\int e^A e^{V(x)} dx$ where A is a quadratic form and $V(x)$ does not contain linear and quadratic terms as a sum of Feynman diagrams:

$$\int \exp\left(\frac{1}{2} a_{ij} x^i x^j \right) e^{V(x)} dx = e^W \tag{2.11}$$

where W is a sum of connected Feynman diagrams with propagator a^{ij} (inverse to a_{ij}) and with vertices determined by $V(x)$. Using this fact and Fourier transform we obtain a diagram technique for $T_{e,\tilde{e}} e^F$.

It follows from this statement that the action of $T_{e,\tilde{e}}$ on the space of semiclassical wave functions is given by rational expressions. This means that the action can be defined over an arbitrary field (in particular, over p-adic numbers).

The above statements can be reformulated in the language of representation theory. Assigning to every symplectic transformation (2.2) a unitary operator T defined by (2.4) we obtain a multivalued representation of the symplectic group $Sp(n, \mathbb{R})$ and corresponding Lie algebra $sp(n, \mathbb{R})$ (metaplectic representation). The representation of the Lie algebra $sp(n, \mathbb{R})$ can be extended to a representation of its complexification $sp(n, \mathbb{C})$ in an obvious way. However, the metaplectic representation of $Sp(n, \mathbb{R})$ cannot be extended to a representation of $Sp(n, \mathbb{C})$ because $Sp(n, \mathbb{C})$ is simply connected and therefore it does not have any non-trivial multivalued representations. (See Deligne [4] for more detailed analysis.)

3. B-model

From the mathematical viewpoint, the genus 0 B-model on a compact Calabi-Yau threefold X is a theory of variations of complex structures on X. Let us denote by $\mathcal{M}$ the moduli space of complex structures on X. For every complex structure, we have a non-vanishing holomorphic $(3, 0)$-form Ω on X, defined up to a constant factor. Assigning the set of forms $\lambda\Omega$ to every complex structure we obtain a line bundle $\mathcal{L}$ over $\mathcal{M}$. The total space of this bundle, i.e. the space of all pairs (complex structure on X, form $\lambda\Omega$), will be denoted by $\widetilde{\mathcal{M}}$. Every form Ω specifies an element of $H^3(X, \mathbb{C})$ (middle-dimensional cohomology of X) that will be denoted by the same symbol. Notice that Ω depends on the complex structure on X, but $H^3(X, \mathbb{C})$ does not depend on complex structure. More precisely, the groups $H^3(X, \mathbb{C})$ form a vector bundle over $\mathcal{M}$ and this bundle is equipped with a flat connection $\boldsymbol{\partial}_a$ (Gauss-Manin connection). In other words, the groups $H^3(X, \mathbb{C})$ where X runs over small open subset of $\mathcal{M}$ are canonically isomorphic. However, the bundle at hand is not necessarily trivial: the Gauss-Manin connection can have non-trivial monodromies. Going around a closed homotopically non-trivial loop γ in $\mathcal{M}$, we obtain a (possibly) non-trivial isomorphism $M_\gamma : H^3(X, \mathbb{C}) \to H^3(X, \mathbb{C})$. The set of all elements of $H^3(X, \mathbb{C})$ corresponding to forms Ω constitutes a lagrangian submanifold L of $H^3(X, \mathbb{C})$. (The cup product on $H^3(X, \mathbb{C})$ taking values in $H^6(X, \mathbb{C}) = \mathbb{C}$ specifies a symplectic structure on $H^3(X, \mathbb{C})$. The fact that L is lagrangian follows immediately from the Griffiths transversality.) We can also say that we have a family of lagrangian submanifolds $L_\tau \subset H^3(X_\tau, \mathbb{C})$ where $H^3(X_\tau, \mathbb{C})$ denotes the third cohomology of the manifold X equipped with the complex structure $\tau \in \mathcal{M}$. Notice that the Lagrangian submanifold L is invariant with respect to the monodromy group (the group of monodromy transformations M_γ).

The B-model on X for an arbitrary genus can be obtained by means of quantization of genus 0 theory, the role of the Planck constant is played by λ^2, where λ is the string coupling constant. (More precisely, we should talk about B-model coupled to gravity or about B-model topological string.) Let us fix a symplectic basis $\{e_A, e^A\}$ in the vector space $H^3(X_\tau, \mathbb{C})$. Every element $\omega \in H^3(X_\tau, \mathbb{C})$ can be represented in the form $\omega = x^A e_A + x_A e^A$, where the coordinates x^A, x_A can be represented as $x^A =< e^A, \omega >, x_A = - < e_A, \omega >$. Quantizing the symplectic vector space $H^3(X_\tau, \mathbb{C})$ by means of polarization $\{e_A, e^A\}$, we obtain a vector bundle $\mathcal{H}$ with fibers $\mathcal{H}_e$. (It would be more precise to denote the fiber by $\mathcal{H}_{\tau,e}$ stressing that a point of the base of the bundle $\mathcal{H}$ is a pair (τ, e) where $\tau \in \mathcal{M}$ and e is a symplectic basis in $H^3(X_\tau, \mathbb{C})$, however, we will use the notation $\mathcal{H}_e$, having in mind that the notation e for the basis already includes the information about the corresponding point $\tau = \tau(e)$ of the moduli space $\mathcal{M}$.) As usual, we have a projectively flat connection on the bundle $\mathcal{H}$. Let us denote by $\mathcal{B}$ the space of all symplectic bases in the cohomology $H^3(X_\tau, \mathbb{C})$ where τ runs over the moduli space $\mathcal{M}$. Then the total space of the bundle $\mathcal{H}$ can be identified with the direct product $\mathcal{B} \times \mathcal{E}$, where $\mathcal{E}$ stands for the space of functions depending on x^A. Let us suppose that the basis $\{e_A, e^A\}$ depends on the parameters $\sigma^1, \cdots, \sigma^K$ and

$$\begin{aligned} \partial_i e^A &= m^A_B e^B + n^{AB} e_B \\ \partial_i e_A &= r_{AB} e^B + s^B_A e_B, \end{aligned} \tag{3.1}$$

where in the calculation of the derivatives $\partial_i = \frac{\partial}{\partial \sigma^i}$, we identify the fibers $\mathcal{H}_e$ by means of Gauss-Manin connection. Then a projectively flat section $\Psi(x^A, \sigma^i, \lambda)$ satisfies the following equation

$$\frac{\partial \Psi}{\partial \sigma^i} = \left[-\frac{1}{2}\lambda^{-2} r_{AB} x^A x^B + m^A_B x^B \frac{\partial}{\partial x^A} + \frac{1}{2}\lambda^2 n^{BC} \frac{\partial^2}{\partial x^B \partial x^C} + C_i(\sigma) \right] \Psi(x^A, \sigma^i, \lambda). \tag{3.2}$$

This follows immediately from Equation (2.7). (Recall that the wave function Ψ depends on half of coordinates on the symplectic basis $\{e_A, e^A\}$.)

The wave function of the B-model topological string is a projectively flat section Ψ of the bundle $\mathcal{H}$ that in semiclassical approximation corresponds to the lagrangian submanifold L coming from genus 0 theory. Of course, such a section is not unique and one needs additional assumptions to determine the wave function.

Notice that the right object to consider in B-model is the wave function $\Psi(x, e, \lambda)$ defined as a function of x^A and polarization $e = \{e_A, e^A\}$. However, it is convenient to work with Ψ restricted to certain subset of the set of polarizations. In particular, we can fix an integral basis $\{g_A(\tau), g^A(\tau)\}$ in $H^3(X_\tau, \mathbb{C})$ that varies continuously with $\tau \in \mathcal{M}$. (The integral vectors of $H^3(X_\tau, \mathbb{C})$ are defined as vectors in the image of integral cohomology $H^3(X_\tau, \mathbb{Z})$ in $H^3(X_\tau, \mathbb{C})$.) It is obvious that the vectors $\{g_A, g^A\}$ are covariantly constant with respect the Gauss-Manin connection, therefore we can assume that in this polarization the wave function does not depend on the point of moduli space. It can be represented in the form

$$\Psi_{\text{real}}(x^A, \lambda) = \exp\left[\sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g(x^A)\right], \tag{3.3}$$

where $\mathcal{F}_g$ is the contribution of genus g surfaces. The leading term in the exponential as always specifies the semiclassical approximation; it corresponds to the genus zero free energy $F_0(x^A)$. In the next section, we will calculate the transformation of the wave function Ψ from the real polarization to some other polarizations. It is important to emphasize that the Gauss-Manin connection can have non-trivial monodromies, hence the integral basis $\{g_A(\tau), g^A(\tau)\}$ is a multivalued function of $\tau \in \mathcal{M}$. The quantum state represented by the wave function Ψ_{real} should be invariant with respect to the monodromy transformation M_γ; in other words, one can find such numbers c_γ that

$$\widetilde{M_\gamma}\Psi_{\text{real}} = c_\gamma \Psi_{\text{real}}, \tag{3.4}$$

where $\widetilde{M_\gamma}$ stands for the transformation of wave function corresponding to the symplectic transformation M_γ (in other words $\widetilde{M_\gamma}$ corresponds to M_γ under metaplectic representation). The condition (3.4) imposes severe restrictions on the state Ψ_{real}, but it does not determine Ψ_{real} uniquely.

In the next section, we will calculate the transformation of the wave function Ψ from the real polarization to some other polarizations.

4. Complex hermitian polarization

Let us introduce special coordinates on $\mathcal{M}$ and $\widetilde{\mathcal{M}}$. We fix an integral symplectic basis g^0, g^a, g_a, g_0 in $H^3(X, \mathbb{C})$. (This means that the vectors of symplectic basis g^A, g_A belong to the image of cohomology with integral coefficients $H^3(X, \mathbb{Z})$ in $H^3(X, \mathbb{C})$. We use small Roman letters for indices running over the set $\{1, 2, \cdots, r = h^{2,1}\}$ and capital Roman letters for indices running over the set $\{0, 1, \cdots, r = h^{2,1}\}$.) Then special coordinates of $\widetilde{\mathcal{M}}$ are defined by the formula

$$X^A =< g^A, \Omega > .$$

Recall that $\dim_{\mathbb{C}} \mathcal{M} = h^{2,1}$, $\dim_{\mathbb{C}} \widetilde{\mathcal{M}} = \dim_{\mathbb{C}} \mathcal{M} + 1 = \frac{1}{2} \dim_{\mathbb{C}} H^3(X, \mathbb{C})$. Hence, we have the right number of coordinates.

The functions $x^A =< g^A, >$ and $x_A = - < g_A, >$ define symplectic coordinates on $H^3(X, \mathbb{C})$; on the lagrangian submanifold L we have

$$x_A = \frac{\partial F_0(x^A)}{\partial x^A},$$

where the function F_0 (the generating function of the lagrangian submanifold L) has the physical meaning of genus 0 free energy. Notice that the lagrangian submanifold L is invariant respect to dilations (this is a consequence of the fact that Ω is defined only up to a factor), and it follows that F_0 is a homogeneous function of degree 2.

Identifying $\widetilde{\mathcal{M}}$ with the lagrangian submanifold L we see that the functions x^A on L are special coordinates on $\widetilde{\mathcal{M}}$.

If two points of $\widetilde{\mathcal{M}}$ correspond to the same point of $\mathcal{M}$ (to the same complex structure), then the forms Ω are proportional; the same is true for the special coordinates X^A. This means that X^A can be regarded as homogeneous coordinates on $\mathcal{M}$. We can construct inhomogeneous coordinates $t^1, \cdots, t^r$ by taking $t^i = \frac{X^i}{X^0}$, $i = 1, \cdots, h^{2,1}$. One can consider the free energy as a function $f_0(t^1, \cdots, t^r)$; then

$$F_0(X^0, \cdots, X^r) = (X^0)^2 f_0\left(\frac{X^1}{X^0}, \cdots, \frac{X^r}{X^0}\right).$$

Let us work with the special coordinates $X^A =< g^A, \Omega >$ on $\widetilde{\mathcal{M}}$ and coordinates $t^a = \frac{X^a}{X^0}$ on $\mathcal{M}$. One can say that we are working with homogeneous coordinates $X^0, \cdots, X^r$

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assuming that $X^0 = 1$. We define cohomology classes Ω_a on a Calabi-Yau manifold X using the formula

$$\Omega_a = \boldsymbol{\partial}_a\Omega + \omega_a\Omega,$$

where ω_a are determined from the condition $\Omega_a \in H^{2,1}$ and $\boldsymbol{\partial}_a$ $(a = 1, \cdots, r = h^{2,1})$ stands for the Gauss-Manin covariant derivatives with respect to the special coordinates $t^a = \frac{X^a}{X^0}$ on $\mathcal{M}$. Representing Ω as $X^A g_A + \partial_A F_0 g^A$ and taking into the account that g^A and g_A are covariantly constant, we obtain

$$\boldsymbol{\partial}_a\Omega = g_a + \partial_a\partial_B F_0 g^B = g_A + \tau_{aB}g^B.$$

(Recall that it follows from the Griffiths transversality that $\boldsymbol{\partial}_a\Omega \in H^{3,0} + H^{2,1}$. Every element of $H^{3,0}$ is represented in the form $\omega\Omega$; this follows from the relation $H^{3,0} = \mathbb{C}$.)

Now we can define a basis of $H^3(X, \mathbb{C})$ consisting of vectors $(\Omega, \Omega_a, \overline{\Omega}_a, \overline{\Omega})$. (It is obvious that (Ω, Ω_a) span $H^{3,0} + H^{2,1}$. Similarly, $\overline{\Omega}_a, \overline{\Omega}$ span $H^{2,1} + H^{0,3}$.) Let us introduce the notation

$$e^{-K} = -i < \Omega, \overline{\Omega} >= i\left(\overline{X}^A \frac{\partial F_0}{\partial X^A} - X^A \overline{\frac{\partial F_0}{\partial X^A}}\right). \tag{4.1}$$

(The function K can be considered as a potential of a Kähler metric on $\mathcal{M}$.) Then we can calculate Ω_a using the relation that $< \Omega_a, \overline{\Omega} >= 0$; we obtain

$$\Omega_a = \boldsymbol{\partial}_a\Omega - \partial_a K\Omega.$$

We can relate the basis $\{\Omega, \Omega_a, \overline{\Omega}_a, \overline{\Omega}\}$ to the integral symplectic basis $\{g^A, g_A\}$ by the following formulas

$$\Omega = X^A g_A + \frac{\partial F_0}{\partial X^A} g^A, \qquad \Omega_a = g_a + \tfrac{\partial^2 F_0}{\partial X^a \partial X^B} g^B - \partial_a K(X^A g_A + \tfrac{\partial F_0}{\partial X^A} g^A), \tag{4.2}$$

$$\overline{\Omega} = \overline{X}^A g_A + \overline{\frac{\partial F_0}{\partial X^A}} g^A, \qquad \overline{\Omega}_a = g_a + \overline{\tfrac{\partial^2 F_0}{\partial X^a \partial X^B}} g^B - \partial_a K(\overline{X^A} g_A + \overline{\tfrac{\partial F_0}{\partial X^A}} g^A). \tag{4.3}$$

The symplectic pairings between $\{\Omega, \Omega_i, \overline{\Omega}_i, \overline{\Omega}\}$ are

$$< \overline{\Omega}, \Omega >= -ie^{-K}, \qquad < \overline{\Omega}_i, \Omega_j >= -iG_{\bar{i}j}e^{-K},$$

where $G_{i\bar{j}}$ is a Kähler metric on $\mathcal{M}$ defined by $G_{i\bar{j}} = \overline{\partial}_j\partial_i K$. As the commutation relations are not the standard one, we introduce the following cohomology classes

$$\widetilde{\Omega}^i = iG^{i\bar{j}}e^K\overline{\Omega}_j, \qquad \widetilde{\Omega} = ie^K\overline{\Omega}.$$

Due to the relations

$$< \widetilde{\Omega}, \Omega >= 1, \qquad < \widetilde{\Omega}^i, \Omega_j >= \delta_i^j,$$

we can say that $\{\Omega, \Omega_a, \widetilde{\Omega}^a, \widetilde{\Omega}\}$ constitutes a symplectic basis, which specifies a complex hermitian polarization.

Directly differentiating the above expressions with respect to the parameters t^a and $\bar{t}^a$, we have

$$\begin{aligned} \boldsymbol{\partial}_i\Omega &= \Omega_i - \partial_i K\Omega, & \boldsymbol{\partial}_i\Omega_j &= -\partial_i K\Omega_j + \Gamma^k_{ij}\Omega_k + iC_{ijk}\widetilde{\Omega}^k, \\ \boldsymbol{\partial}_i\widetilde{\Omega} &= \partial_i K\widetilde{\Omega}, & \boldsymbol{\partial}_i\widetilde{\Omega}^j &= \partial_i K\widetilde{\Omega}^j - \sum_k \Gamma^k_{ij}\widetilde{\Omega}^k - \widetilde{\Omega}\delta_{ij}; \end{aligned} \tag{4.4}$$

where Γ^k_{ij} is the Christoeffel symbol for the Kähler metric $G_{i\bar{j}}$. And

$$\begin{aligned} \overline{\boldsymbol{\partial}}_i \Omega &= 0, & \overline{\boldsymbol{\partial}}_i \Omega_j &= G_{\bar{i}j}\Omega, \\ \overline{\boldsymbol{\partial}}_i \widetilde{\Omega} &= -G_{\bar{i}j}\widetilde{\Omega}^j, & \overline{\boldsymbol{\partial}}_i \widetilde{\Omega}^j &= -ie^{2K}\overline{C}_{\bar{i}}^{jk}\Omega_k. \end{aligned} \tag{4.5}$$

Applying (3.2), we obtain from (4.4), (4.5) equations governing the dependence of the state $\Psi(x^I, t^i, \bar{t}^i, \lambda)$ on $t^i, \bar{t}^i$ (holomorphic anomaly equation).

$$\frac{\partial\Psi}{\partial\bar{t}^i} = \left[\frac{1}{2}\lambda^2 e^{2K}\overline{C}_{\bar{i}\bar{j}\bar{k}}G^{\bar{j}j}G^{\bar{k}k}\frac{\partial^2}{\partial x^i \partial x^j} + G_{\bar{i}j}x^j\frac{\partial}{\partial x^0} + C_i\right]\Psi, \tag{4.6}$$

$$\frac{\partial\Psi}{\partial t^i} = \left[x^0\frac{\partial}{\partial x^i} - \partial_i K\left(x^0\frac{\partial}{\partial x^0} + x^j\frac{\partial}{\partial x^j}\right) - \Gamma^k_{ij}x^j\frac{\partial}{\partial x^k} - \frac{1}{2}\lambda^{-2}C_{ijk}x^j x^k + D_i\right]\Psi. \tag{4.7}$$

Notice that usually these equations are written with $C_i = 0, D_i = 0$. This is possible if we consider only one of these equations; fixing C_i or D_i corresponds to (physically irrelevant) choice of normalization of the wave function. However, in general it is impossible to assume that $C_i = 0, D_i = 0$. (We can eliminate C_i or D_i changing the normalization of the wave function, but we cannot eliminate both of them.) Let us emphasize that C_i and D_i are constraint by the requirement that (4.6) has a solution.

5. Holomorphic polarization

Let us start again with the integral symplectic basis $\{g_0, g_a, g^a, g^0\}$. We will normalize the form Ω requiring that $< g^0, \Omega >= X^0 = 1$. We would like to define a symplectic basis in the middle dimensional cohomology that depends holomorphically on the points of moduli space. Namely, we will consider the following basis in $H^3(X, \mathbb{C})$,

$$\begin{aligned} e^0 &= g^0, \\ e^a &= g^a - t^a g^0, \\ e_a &= \boldsymbol{\partial}_a \Omega, \\ e_0 &= \Omega = g_0 + X^a g_a + \partial_a F_0 g^a + \partial_0 F_0 g^0, \end{aligned}$$

where $\boldsymbol{\partial}_a$ stands for the Gauss-Manin connection. It is easy to check that this basis is symplectic.

Using the above relations, we obtain an expression of the new basis in terms of the integral symplectic basis g^A, g_A,

$$\begin{aligned} e^0 &= g^0 \\ e^a &= g^a - t^a g^0, \\ e_0 &= g_0 + t^a g_a + \frac{\partial f_0}{\partial t^a}g^a + \left(2f_0 - t^a\frac{\partial f_0}{\partial t^a}\right)g^0, \\ e_a &= g_a + \frac{\partial^2 f_0}{\partial t^a \partial t^b}g^b + \left(\frac{\partial f_0}{\partial t^a} - t^b\frac{\partial^2 f_0}{\partial t^a \partial t^b}\right)g^0. \end{aligned} \tag{5.1}$$

Let q_A, q^A denote the coordinates on the basis g^A, g_A, and ϵ_A, ϵ^A the coordinates on the basis e^A, e_A. Equation (2.8) permits us to relate the wave function in real polarization to the wave function in our new basis (in holomorphic polarization)

$$\Psi_{\mathrm{hol}}(\epsilon^I, t^i, \lambda) = e^{-\frac{1}{2}\lambda^{-2}R_{AB}\epsilon^A\epsilon^B}\Psi_{\mathrm{real}}(\epsilon^0, \epsilon^i + t^i\epsilon^0, \lambda), \tag{5.2}$$

where R is a the matrix

$$\begin{pmatrix} 2f_0 & \frac{\partial f_0}{\partial t^a} \\ \frac{\partial f_0}{\partial t^a} & \frac{\partial^2 f_0}{\partial t^a \partial t^b} \end{pmatrix}.$$

Notice that Ψ_{hol} is defined up to a t-dependent factor; we use Equation (5.2) to fix this factor.

Using that g^0, g^a, g_a, g_0 are covariantly constant with respect to the Gauss-Manin connection $\boldsymbol{\partial}_a$, we see that

$$\begin{aligned} \boldsymbol{\partial}_b e_0 &= e_b, & \boldsymbol{\partial}_b e_a &= C_{abc}e^c \\ \boldsymbol{\partial}_b e^a &= \delta_{ab}e^0, & \boldsymbol{\partial}_b e^0 &= 0, \end{aligned} \tag{5.3}$$

where $C_{abc} = \partial_a\partial_b\partial_c f_0$. Applying Equation (3.2), we obtain from Equation (5.3) the dependence of the state $\Psi_{\mathrm{hol}}(\epsilon^A, t^i, \lambda)$ on the coordinates $t^1, \cdots, t^h$

$$\frac{\partial \Psi_{\mathrm{hol}}(\epsilon^A, t^i, \lambda)}{\partial t^a} = \left(\epsilon^0 \frac{\partial}{\partial \epsilon^a} - \frac{1}{2}\lambda^{-2}C_{abc}\epsilon^b\epsilon^c + \sigma_a(t)\right)\Psi_{\mathrm{hol}}(\epsilon^A, t^i, \lambda). \tag{5.4}$$

The function Ψ_{hol} defined by the Equation (5.2) obeys Equation (5.4) with $\sigma_a = 0$. We remark that because our basis is holomorphic, the state Ψ does not depend on antiholomorophic variables $\bar{t}^i$. Therefore Equation (5.4) is the only equation the state $\Psi_{\mathrm{hol}}(\epsilon_i, t^i, \lambda)$ has to satisfy. This equation can be easily solved. The solution can be written as follows,

$$\begin{aligned} \Psi &= \exp(W_1 + W_2), \\ W_1 &= W(\epsilon^0, \epsilon^0 t^a + \epsilon^a), \\ W_2 &= -\lambda^{-2}\left(\frac{1}{2}\frac{\partial^a f_0}{\partial t^i \partial t^j}\epsilon^i\epsilon^j + \frac{\partial f_0}{\partial t^i}\epsilon^i\epsilon^0 + f_0(\epsilon^0)^2\right), \end{aligned} \tag{5.5}$$

here W is an arbitrary function of $h^{2,1}+1$ variables.

Comparing the above expression with Equation (5.2), we obtain

$$\exp(W) = \Psi_{\mathrm{real}}. \tag{5.6}$$

Let us consider now the B-model in the neighborhood of the maximally unipotent boundary point. We choose g^0 as covariantly constant cohomology class that can be extended to the boundary point and we define g^a as covariantly constant cohomology classes having logarithmic singularities at the boundary point. The special coordinates coincide with the canonical coordinates and the basis $\{e^A, e_A\}$ coincides with the basis that is widely used in the theory of mirror symmetry. (See [3], section 6.3). This can be derived, for example, from the fact that the Gauss-Manin connection described by the formula (5.3) has the same form in both bases.

6. Partition function of B-model

The partition function Ψ of the topological sigma model on a Calabi-Yau threefold X (and more generally of twisted $N = 2$ superconformal theory) can be represented as $\Psi = e^F$, where

$$F = \sum_g \lambda^{2g-2} F_g(t, \bar{t}), \tag{6.1}$$

and F_g has a meaning of contribution of surfaces of genus g to the free energy. The correlation functions $C^{(g)}_{i_1,\cdots,i_n}$ can be obtained from F_g by means of covariant differentiation. Notice that in Equation (6.1) we can consider t and $\bar{t}$ as independent complex variables. The covariant derivatives with respect to t coincide with $\frac{\partial}{\partial t^i}$ in the limit when $\bar{t} \to \infty$ and t remains finite.

It is convenient to introduce the generating functional of correlation functions

$$W(\lambda, x, t, \bar{t}) = \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^{2g-2} C^{(g)}_{i_1,\cdots,i_n} x^{i_1} \cdots x^{i_n} + \left(\frac{\chi}{24} - 1\right) \log(\lambda), \tag{6.2}$$

where $C^{(g)}_{i_1,\cdots,i_n} = 0$ for $2g - 2 + n \leq 0$. The number χ is defined as the difference between the numbers of the bosonic and fermionic modes; in the case of topological sigma-model it coincides with the Euler characteristic of X (up to a sign).

The function W obeys the following holomorphic anomaly equations (Equation 3.17, 3.18, [2])

$$\frac{\partial}{\partial \bar{t}^i} \exp(W) = \left[\frac{\lambda^2}{2} \overline{C}_{ijk} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \left(\lambda \frac{\partial}{\partial \lambda} + x^k \frac{\partial}{\partial x^k}\right)\right] \exp(W), \tag{6.3}$$

and

$$\left[\frac{\partial}{\partial t^i} + \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k} + \partial_i K \left(\frac{\chi}{24} - 1 - \lambda \frac{\partial}{\partial \lambda}\right)\right] \exp(W) = \left[\frac{\partial}{\partial x^i} - \partial_i F_1 - \frac{1}{2\lambda^2} C_{ijk} x^j x^k\right] \exp(W). \tag{6.4}$$

One can modify the definition of W by introducing a new function $\widetilde{W}$,

$$\begin{aligned} \widetilde{W}(\lambda, x^i, \rho, t, \bar{t}) &= \sum_{g=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^{2g-2} C^{(g)}_{i_1,\cdots,i_n} x^{i_1} \cdots x^{i_n} \rho^{-n-(2g-2)} + \left(\frac{\chi}{24} - 1\right) \log \rho \\ &= W\left(\frac{\lambda}{\rho}, \frac{x}{\rho}, t, \bar{t}\right) - \left(\frac{\chi}{24} - 1\right) \log(\lambda). \end{aligned} \tag{6.5}$$

The function $\widetilde{W}$ (we will call it BCOV wave function) satisfies the equations

$$\frac{\partial}{\partial \bar{t}^i} \exp(\widetilde{W}) = \left[\frac{\lambda^2}{2} \overline{C}^{jk}_{\bar{i}} \frac{\partial^2}{\partial x^j \partial x^k} + G_{\bar{i}j} x^j \frac{\partial}{\partial \rho}\right] \exp(\widetilde{W}), \tag{6.6}$$

$$\begin{aligned} \frac{\partial}{\partial t^i} \exp(\widetilde{W}) = \Bigg[&\rho \frac{\partial}{\partial x^i} - \partial_i K \left(\rho \frac{\partial}{\partial \rho} + x^j \frac{\partial}{\partial x^j}\right) - \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k} - \frac{1}{2\lambda^2} C_{ijk} x^j x^k \\ &- \partial_i F_1 - \partial_i K \left(\frac{\chi}{24} - 1\right)\Bigg] \exp(\widetilde{W}). \end{aligned} \tag{6.7}$$

Equations (6.6) follows from (6.3); it is equivalent to the equation (6.11) of [2]. And equation (6.7) follows from equation (6.4).

The above equations are valid for any topologically twisted $N = 2$ superconformal theory coupled to gravity. We will apply them to the study of B-model. In this case, it is clear from comparison of equations (6.6) with (4.6) that $\exp(\widetilde{W})$ can be interpreted as a wave function corresponding to the complex hermitian polarization considered in section 3. The equation (6.6) implies that $\exp(\widetilde{W})$ is a projectively flat section. Let us emphasize that $\exp(\widetilde{W})$ is a well defined function determined from worldsheet considerations and the wave function is specified only up to a factor. Considering $\exp(\widetilde{W})$ as a wave function we fix this factor. The fact that the wave function in complex hermitian polarization can be considered as a one-valued function on the whole moduli space of complex structures (monodromy transformations act trivially) was essentially used in [1]. Notice also that the worldsheet interpretation of the wave function permits us to analyze its behavior at boundary points of the moduli space; this information imposes further restrictions on the quantum state obtained by quantization of genus zero theory.

The function $\widetilde{W}(\lambda, x, \rho, t, \infty)$ can be represented in terms of $F(\lambda, t, \infty)$ in the following way

$$\begin{aligned}\widetilde{W}(\lambda, x, \rho, t, \infty) &= \sum_g \frac{1}{n!}\left(\frac{\lambda}{\rho}\right)^{2g-2} F_g\left(t + \frac{x}{\rho}, \infty\right) - F_1(t,\infty) - \left(\frac{\lambda}{\rho}\right)^{-2}\Bigg(F_0(t,\infty) \\ &\quad + \frac{\partial F_0(t,\infty)}{\partial t^i}\cdot\frac{x^i}{\rho} + \frac{1}{2}\frac{\partial^2 F_0(t,\infty)}{\partial t^i \partial t^j}\cdot\frac{x^i x^j}{\rho^2}\Bigg) - \left(\frac{\chi}{24} - 1\right)\log\rho \\ &= F\left(\frac{\lambda}{\rho}, t + \frac{x}{\rho}, \infty\right) - F_1(t,\infty) - \left(\frac{\lambda}{\rho}\right)^{-2}\Bigg(F_0(t,\infty) + \\ &\quad \frac{\partial F_0(t,\infty)}{\partial t^i}\cdot\frac{x^i}{\rho} + \frac{1}{2}\frac{\partial^2 F_0(t,\infty)}{\partial t^i \partial t^j}\cdot\frac{x^i x^j}{\rho^2}\Bigg) - \left(\frac{\chi}{24} - 1\right)\log\rho \end{aligned} \tag{6.8}$$

(Due to the relation $C^{(g)}_{i_1,\cdots,i_n} = \partial_{i_1}\cdots\partial_{i_n} F_g(\lambda, t, \infty)$, the expression for $\widetilde{W}$ as $\bar{t} \to \infty$ can be considered as Taylor series; for $g = 0$ and $g = 1$ the first few terms of the Taylor series are missing because one assumes that $C^{(g)}_{i_1,\cdots,i_n} = 0$ for $2g - 2 + n \le 0$.) Notice that the above formula can be used both for A-model and for B-model (in the latter case t^a are canonical coordinates in the neighborhood of maximally unipotent boundary point).

As we have seen in the language of B-model, $\exp(\widetilde{W}(\lambda, x, \rho, t, \infty))$ can be interpreted as a wave function in the complex hermitian polarization for $\bar{t} = \infty$. From the other side, the complex hermitian polarization for $\bar{t} = \infty$ coincides with holomorphic polarization (see appendix). This means that up to a t-dependent factor $\exp(\widetilde{W}(\lambda, x, \rho, t, \infty))$ coincides with the wave function in holomorphic polarization.

Let us give another proof of this fact that permits us to calculate the t-dependent factor.

We defined the wave function in holomorphic polarization as a solution to the equation (5.4). The function $\exp(\widetilde{W}(\lambda, x, \rho, t, \infty))$ obeys a little bit different equation (for $\bar{t} = \infty$, we can take $\partial_a K = 0$, $\Gamma^k_{aj} = 0$ in the equation (6.6)). Comparing these equations, we see that (up to a constant factor)

$$\Psi(\lambda, x, \rho, t) = \exp(\widetilde{W}(\lambda, x, t, \rho, \infty)) \exp(F_1). \tag{6.9}$$

It follows from this expression that our function $\Psi(\lambda, x, \rho, t)$ coincides with the modification of BCOV wave function $\exp(\widetilde{W}(\lambda, x, t, \rho, \infty))$ considered in [11], [6].

Using the expression (6.8) of $\widetilde{W}$ and (6.9), we have the following expression for Ψ

$$\begin{aligned}\Psi = \exp\Bigg(F\left(\frac{\lambda}{\rho}, t + \frac{x}{\rho}, \infty\right) - \lambda^{-2}\left(F_0(t,\infty)\rho^2 + \frac{\partial F_0}{\partial t^i}x^i\rho + \frac{1}{2}\frac{\partial^2 F_0}{\partial t^i \partial t^j}x^i x^j\right) \\ -\left(\frac{\chi}{24} - 1\right)\log(\rho)\Bigg).\end{aligned} \tag{6.10}$$

We use Equation (5.2) to compute the corresponding wave function in the real polarization by identifying $\rho = \epsilon^0, x^i = \epsilon^i$.

$$\Psi_{\text{real}}(\epsilon^0, \epsilon^i + t^i\epsilon^0, \lambda) = \exp\left(F\left(\frac{\lambda}{\epsilon^0}, t^i + \frac{\epsilon^i}{\epsilon^0}, \infty\right) - \left(\frac{\chi}{24} - 1\right)\log(\epsilon^0)\right). \tag{6.11}$$

Accordingly, if we set $x^0 = \epsilon^0$ and $x^i = \epsilon^i + t^i\epsilon^0$, we have

$$\Psi_{\text{real}}(x^0, x^i, \lambda) = \exp\left(F\left(\frac{\lambda}{x^0}, \frac{x^i}{x^0}, \infty\right) - \left(\frac{\chi}{24} - 1\right)\log(x^0)\right). \tag{6.12}$$

It follows from this equation that

$$\Psi_{\text{real}}(cx^0, cx^i, c\lambda) = \Psi_{\text{real}}(x^0, x^i, \lambda)c^{-(\frac{\chi}{24}-1)} \tag{6.13}$$

Using (2.10) one can conclude that similar homogeneity property is valid in any polarization:

$$\Psi(cx, e, c\lambda) = \Psi(x, e, \lambda)c^{-(\frac{\chi}{24}-1)} \tag{6.14}$$

To clarify the physical meaning of $\exp(\widetilde{W})$, it is convenient to consider the mirror A-model and to take the limit $\bar{t} \to \infty$. Then the free energy F_g and therefore the functions W and $\widetilde{W}$ can be expressed in terms of Gopakumar-Vafa invariants n_β^g and topological invariants of the mirror manifold $\widetilde{X}$. Namely

$$F = ln\Psi = \sum_g \lambda^{2g-2}F_g(t) = F' + F'' \tag{6.15}$$

can be represented as a sum of two summands F' and F'' where F' corresponds to non-trivial instanton contribution of the mirror A-model with the form

$$F' = \sum_{n,g,\beta\neq 0} n_\beta^g \frac{1}{m}\left(2\sin\frac{m\lambda}{2}\right)^{2g-2} e^{nt^\beta}, \tag{6.16}$$

and the constant map contribution F'' can be represented as

$$F'' = const + \lambda^{-2}\sum \frac{t^{\beta_1}t^{\beta_2}t^{\beta_3}}{3!}\int_{\widetilde{X}} \beta_1 \cup \beta_2 \cup \beta_3 - \frac{t^\beta}{24}\int_{\widetilde{X}} \beta \cup c_2(\widetilde{X}). \tag{6.17}$$

In (6.16) we assume that β runs over the two-dimensional integral homology group of $\widetilde{X}$ (more precisely, only the elements in the positive cone of this group are relevant); in (6.17) β runs over a basis of this group. Recall that the two-dimensional cohomology group labels the deformations of Kähler structures on $\widetilde{X}$; in the language of mirror B-model it corresponds to the cohomology group $H^{2,1}(X)$ that labels deformations of complex structures; the coordinates t^β correspond to canonical coordinates on the moduli space of complex structures of the corresponding B-model.

Instead of free energy $F = F' + F''$ one can consider the partition function $Z = e^F$ represented as a product of two factors $Z' = e^{F'}$ and $Z'' = e^{F''}$. By means of formal manipulations (see [7]), one can derive from (6.16) the following expression[2] for Z' :

$$Z' = \prod (1 - \Lambda^s q^\beta)^{m^s_\beta} \tag{6.18}$$

where

$$\begin{aligned} m^s_\beta &= s n^0_\beta + (-1)^{1+s} \sum_{g \geq 1+|s|} n^g_\beta \binom{2g-2}{g-1-s}, && (6.19)\\ \Lambda &= e^{-i\lambda}, && (6.20)\\ q^\beta &= \exp t^\beta. && (6.21) \end{aligned}$$

Using the expression (6.9), we obtain an expression of the wave function in holomorphic polarization:

$$\Psi(\lambda, x, \rho, t) = \Psi' \Psi'',$$

where Ψ' is expressed in terms of Gopakumar-Vafa invariants n^g_β with $g \geq 0$. More precisely,

$$\begin{aligned} \Psi'(\lambda, x, \rho, t) &= \exp\left(\sum_{m, g \geq 1, \beta \neq 0} n^g_\beta \frac{1}{m} \left(2 \sin \frac{m\lambda}{2\rho} \right)^{2g-2} e^{m\left(t^\beta + \frac{x^\beta}{\rho}\right)} \right)\\ &= \prod_{s, \beta \neq 0} \left(1 - e^{-is\frac{\lambda}{\rho} + \left(t^\beta + \frac{x^\beta}{\rho}\right)} \right)^{m^s_\beta}. \end{aligned} \tag{6.22}$$

A. Relation between complex hermitian polarization and holomorphic polarization

Here we relate the holomorphic polarization to the complex hermitian polarization.

Since both bases are expressed in terms of the real basis $\{g_A, g^A\}$, we can compute the expression of $\{\Omega, \Omega_a, \widetilde{\Omega}^a, \widetilde{\Omega}\}$ by $\{e_0, e_a, e^a, e^0\}$ as follows.

$$\begin{aligned} \Omega &= e_0,\\ \Omega_a &= e_a - \partial_a K e_0,\\ \widetilde{\Omega} &= i e^K e_0 + i e^K (\overline{X}^a - X^a) e_a + \partial_a K e^a + e^0,\\ \widetilde{\Omega}^a &= G^{a\bar{b}} \big[- i e^K \overline{\partial_b K} e_0 + i e^K e_b - i e^K (\overline{X}^c - X^c) \overline{\partial_b K} e_c\\ &\qquad + (G_{\bar{b}c} + \partial_b K \overline{\partial_c K} - \partial_c K \overline{\partial_b K}) e^c \big] \end{aligned} \tag{A.1}$$

[2] To give a precise meaning to (6.18) one can consider this expression as an element of Novikov ring with generators q^β and with coefficients in Laurent series with respect to Λ; the generators obey the relation $q^\beta q^{\beta'} = q^{\beta + \beta'}$.

We recall that in the neighborhood of maximally unipotent boundary point, the function F_0 has the following expression

$$F_0(X^0, X^i) = \frac{d_{ijk}X^iX^jX^k}{X^0} + c(X^0)^2 + \sigma(q^i),$$

where σ is a holomorphic function and $q^j \overset{def}{=} \exp(2\pi i t^j)$. The expression $\sigma(q^i)$ is bounded in the neighborhood of the boundary point $q^i = 0$.

In the following computation, we set $X^0 = 1$. Substituting the above expression of F_0 into $K = -\log(i(\overline{X}^A\frac{\partial F_0}{\partial X^A} - X^A\overline{\frac{\partial F_0}{\partial X^A}}))$, we obtain

$$K = -\log\left(i\left(\bar{d}_{ijk}\bar{t}^i\bar{t}^j\bar{t}^j - d_{ijk}t^it^jt^k + 3d_{ijk}\bar{t}^it^jt^k - 3\bar{d}_{ijk}t^i\bar{t}^j\bar{t}^k + 2(c-\bar{c}) + \varphi\right)\right),$$

where φ is equal to

$$-\log(q^i)q^i\frac{\partial\sigma}{\partial q^i} + \log(\bar{q}^i)q^i\frac{\partial\sigma}{\partial q^i} + \log(\bar{q}^i)\bar{q}^i\overline{\frac{\partial\sigma}{\partial q^i}} - \log(q^i)\bar{q}^i\overline{\frac{\partial\sigma}{\partial q^i}}.$$

Differentiating K respect to t^a, we have

$$\partial_a K = \frac{3d_{ajk}t^jt^k - 6d_{aik}\bar{t}^it^k + 3\bar{d}_{ajk}\bar{t}^j\bar{t}^k - \partial_a\varphi}{\bar{d}_{ijk}\bar{t}^i\bar{t}^j\bar{t}^j - d_{ijk}t^it^jt^k + 3d_{ijk}\bar{t}^it^jt^k - 3\bar{d}_{ijk}t^i\bar{t}^j\bar{t}^k + 2(c-\bar{c}) + \varphi}$$

Similarly,

$$\overline{\partial_a K} = \frac{3\bar{d}_{ajk}\bar{t}^j\bar{t}^k - 6\bar{d}_{aik}t^i\bar{t}^k + 3d_{ajk}t^jt^k - \overline{\partial_a\varphi}}{d_{ijk}t^it^jt^j - \bar{d}_{ijk}\bar{t}^i\bar{t}^j\bar{t}^k + 3\bar{d}_{ijk}t^i\bar{t}^j\bar{t}^k - 3d_{ijk}\bar{t}^it^jt^k + 2(\bar{c}-c) + \overline{\varphi}}. \tag{A.2}$$

Taking the derivative of $\partial_a K$ respect to $\bar{t}^b$, we obtain

$$\begin{aligned}\bar{\partial}_b\partial_a K = &\frac{-6d_{bak}t^k + 6\bar{d}_{abk}\bar{t}^k}{\bar{d}_{ijk}\bar{t}^i\bar{t}^j\bar{t}^j - d_{ijk}t^it^jt^k + 3d_{ijk}\bar{t}^it^jt^k - 3\bar{d}_{ijk}t^i\bar{t}^j\bar{t}^k + 2(c-\bar{c}) + \varphi} + \\ &-\frac{\left(3d_{ajk}t^jt^k - 6d_{aik}\bar{t}^it^k + 3\bar{d}_{ajk}\bar{t}^j\bar{t}^k + \partial_a\varphi\right)\left(3\bar{d}_{bjk}\bar{t}^j\bar{t}^k - 6\bar{d}_{bik}t^i\bar{t}^k + 3d_{bjk}t^jt^k\right)}{\left(\bar{d}_{ijk}\bar{t}^i\bar{t}^j\bar{t}^j - d_{ijk}t^it^jt^k + 3d_{ijk}\bar{t}^it^jt^k - 3\bar{d}_{ijk}t^i\bar{t}^j\bar{t}^k + 2(c-\bar{c}) + \varphi\right)^2}\end{aligned} \tag{A.3}$$

Also we have

$$e^K = \frac{1}{e^{-K}} = \frac{1}{i\left(\bar{d}_{ijk}\bar{t}^i\bar{t}^j\bar{t}^j - d_{ijk}t^it^jt^k + 3d_{ijk}\bar{t}^it^jt^k - 3\bar{d}_{ijk}t^i\bar{t}^j\bar{t}^k + 2(c-\bar{c}) + \varphi\right)}.$$

Let us consider t and $\bar{t}$ in these formulas as independent complex variables; then the basis $(\Omega, \Omega_a, \widetilde{\Omega}^a, \widetilde{\Omega})$ is not hermitian anymore. We will check that this basis tends to (e_0, e_a, e^a, e^0) as $\bar{q}$ converges to 0 (and therefore $\bar{t} \to \infty$) with fixed t.

Fixing t and taking $\bar{t} \to \infty$, we have that

$$\varphi \sim O(\bar{t}),\ \partial_a\varphi \sim O(\bar{t}),\ \overline{\partial_a\varphi} \sim O(1),$$

where $O(1)$ stands for bounded terms.

And therefore, we have the following asymptotic leading terms.

1. $\partial_a K \sim 3\frac{\bar{d}_{ajk}\overline{t}^j\overline{t}^k}{\bar{d}_{ijk}\overline{t}^i\overline{t}^j\overline{t}^k}$;

2. $\overline{\partial_a K} \sim -3\frac{\bar{d}_{ajk}\overline{t}^j\overline{t}^k}{\bar{d}_{ijk}\overline{t}^i\overline{t}^j\overline{t}^k}$;

3. $G_{a\bar{b}} \sim \frac{6\bar{d}_{abk}\overline{t}^k}{\bar{d}_{ijk}\overline{t}^i\overline{t}^j\overline{t}^k} - \frac{9\bar{d}_{ajk}\bar{d}_{blm}\overline{t}^j\overline{t}^k\overline{t}^l\overline{t}^m}{(\bar{d}_{ijk}\overline{t}^i\overline{t}^j\overline{t}^k)^2}$

4. $e^K \sim \frac{1}{i\bar{d}_{ijk}\overline{t}^i\overline{t}^j\overline{t}^k}$;

5. $\partial_a K\overline{\partial_b K} - \partial_b K\overline{\partial_a K} \sim \frac{\begin{pmatrix} d_{iaj}\bar{d}_{bkl}\overline{X}^i\overline{t}^k\overline{t}^l t^j + \bar{d}_{aij}\bar{d}_{kbl}\overline{t}^i\overline{t}^j\overline{t}^l t^k \\ -d_{ibj}\bar{d}_{akl}\overline{t}^i\overline{t}^k\overline{t}^l t^j - \bar{d}_{bij}\bar{d}_{kal}\overline{t}^i\overline{t}^j\overline{t}^l t^k \end{pmatrix}}{\bar{d}_{ijk}d_{lmn}\overline{t}^i\overline{t}^j\overline{t}^k\overline{t}^l\overline{t}^m\overline{t}^n}$.

If we let $\overline{t}^i = \bar{s}^i\nu$ and $\nu \to \infty$, we have the following asymptotic behavior (up to a factor),

1. $\partial_a K \sim \frac{1}{\nu}$;

2. $\overline{\partial_a K} \sim \frac{1}{\nu}$;

3. $G_{a\bar{b}} \sim \frac{1}{\lambda^2}L_{a\bar{b}}$, where $L_{a\bar{b}}$ is a nondegenerate anti-holomorphic matrix;

4. $e^K \sim \frac{1}{\nu^3}$;

5. $\partial_a K\overline{\partial_b K} - \partial_b K\overline{\partial_a K} \sim \frac{1}{\nu^3}$.

Substituting the above asymptotic expressions into Equation (A.1) and taking the limit $\bar{t} \to \infty$ by letting $\nu \to \infty$, we have

$$\begin{aligned} \Omega &= e_0, \qquad & \Omega_a &= e_a, \\ \widetilde{\Omega} &= e^0, \qquad & \widetilde{\Omega}^a &= e^a. \end{aligned} \tag{A.4}$$

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